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An investigation of the non-trivial zeros of the Riemann zeta function

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Abstract The zeros of the Riemann zeta function outside the critical strip are the so-called trivial zeros. While many zeros of the Riemann zeta function are located on the critical line $\Re(s) = 1/2$, the non-existence of zeros in the remaining part of the critical strip $\Re(s) \in]0, 1[$ remains to be proven. The Riemann zeta functional leads to a relationship between the zeros of the Riemann zeta function on either sides of the critical line. Given s a complex number and $\bar{s}$ its complex conjugate, if s is a zero of the Riemann zeta function in the critical strip $\Re(s) \in]0, 1[$, then we have $\zeta(s) = \zeta(1 - \bar{s})$. As the Riemann hypothesis states that all non-trivial zeros lie on the critical line $\Re(s) = 1/2$, it is enough to show there are no zeros on either sides of the critical line within the critical strip $\Re(s) \in]0, 1[$, to say the Riemann hypothesis is true.

Keywords Riemann zeta function, Riemann hypothesis

1 Introduction

The Riemann zeta function named after Bernhard Riemann (1826 - 1866) is an extension of the zeta function to complex numbers, which primary purpose is the study of the distribution of prime numbers [14]. Although the zeta function which is an infinite series of the inverse power function was studied long time ago and is referred to in a formula known as the Euler product establishing the connection between the zeta function and primes, Riemann's approach is linked with more recent developments in complex analysis at the time of Cauchy. The Riemann hypothesis stating that all non-trivial zeros lie on the critical line $\Re(s) = 1/2$ is crucial for the prime-number theory. Not only the Riemann-von Mangoldt explicit formula for the asymptotic expansion of

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the prime-counting function involves a sum over the non-trivial zeros of the Riemann zeta function, but also the Riemann hypothesis has implications for the accurate estimate of the error involved in the prime-number theorem, which is the description of the asymptotic behavior of the number of primes less than or equal to a real variable.

The Riemann zeta function is a holomorphic function defined as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \tag{1}$$

where s is a complex number and domain of congruence $\Re(s) > 1$. The domain of convergence of the Riemann zeta function is extended to the left of $\Re(s) = 1$ by analytic continuation. The Dirichlet eta function, which is the product of the factor $\left(1 - \frac{2}{2^s}\right)$ with the Riemann zeta function is convergent for $\Re(s) > 0$. This is an alternating series, expressed as follows:

$$\eta(s) = \left(1 - \frac{2}{2^s}\right)\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}\,, \tag{2}$$

where $\Re(s) > 0$, and $\eta(1) = \ln(2)$ by continuity.

The function $\left(1 - \frac{2}{2^s}\right)$ has an infinity of zeros on the line $\Re(s) = 1$ given by $s_k = 1 + \frac{2k\pi i}{\ln 2}$ where $k \in \mathbb{Z}^*$. As $\left(1 - \frac{2}{2^s}\right) = 2 \times \left(2^{\alpha-1}\, e^{i\,\beta \ln 2} - 1\right) / \, 2^{\alpha}\, e^{i\,\beta\ln 2}$, the factor $\left(1 - \frac{2}{2^s}\right)$ has no poles nor zeros in the strip $\Re(s) \in]0, 1[$. It follows that by analytic continuation, the Dirichlet eta function can be used as a proxy for zero finding of the Riemann zeta function in the critical strip $\Re(s) \in]0, 1[$. The Riemann zeta function can be further extended to the left, i.e. $\Re(s) \leqslant 0$ by analytic continuation with the Riemann zeta functional. Whenever the Riemann zeta function is invoked for $\Re(s) \leqslant 1$, it is implicitly referring to its analytic continuation.

2 Elementary propositions

Before delving into core propositions, a prerequisite is the Riemann zeta functional, which is used explicitly in propositions 2, 4 and 9 and implicitly in 5, 6, 7 and 8. The Riemann zeta functional is expressed as follows:

$$\zeta(s) = \mathit{\Pi}(-s)(2\pi)^{s-1} 2 \sin\left(\frac{s\pi}{2}\right)\zeta(1-s)\,. \tag{3}$$

This equation was established by Riemann in 1859. The details of the derivation are provided in [2], p. 13. The standard formulation of (3) is as follows:

$$\zeta(1-s) = \mathit{\Gamma}(s)(2\pi)^{-s} 2 \cos\left(\frac{s\pi}{2}\right)\zeta(s)\,, \tag{4}$$

involves the substitution $s \to 1-s$ and the relationship $\Pi(s-1) = \Gamma(s)$. Let us proceed with key propositions.

Proposition 1 A formula is introduced here for calculations further down, which is expressed as follows:

$$\left[\frac{a^2-b^2}{a^2+b^2} + i\left(\frac{-2\,a\,b}{a^2+b^2}\right)\right][a+i\,b] = a - i\,b\,, \tag{5}$$

where a and b are reals.

Proof By identifying the real and imaginary parts of $(x+i\,y)(a+b\,i) = a-i\,b$, we obtain two equations $a\,x - b\,y = a$ and $b\,x + a\,y = -b$. Eq. (5) is the resolution of these two equations.

Proposition 2 Given s a complex number and $\bar{s}$ its complex conjugate, we have:

$$\zeta(\bar{s}) = \overline{\zeta}(s)\,, \tag{6}$$

where $\overline{\zeta}(s)$ is the complex conjugate of $\zeta(s)$.

Proof Let us say $s = \alpha + i\,\beta$ where α and β are real numbers and $i^2 = -1$. We have:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{[\cos(\beta \ln n) - i\sin(\beta \ln n)]}{n^{\alpha}}\,, \tag{7}$$

where $\Re(s) > 1$.

We have $\frac{1}{n^s} = \frac{1}{n^{\alpha}\exp(\beta i \ln n)} = \frac{1}{n^{\alpha}(\cos(\beta \ln n) + i\sin(\beta \ln n))}$. We then multiply both the numerator and denominator by $\cos(\beta\ln(n)) - i\sin(\beta\ln(n))$. After several simplifications, we get (7). Note that in (7) when β changes its sign, the real part of $\zeta(s)$ remains unchanged, while imaginary part changes sign. This means, the Riemann zeta function has mirror symmetry with respect to the real axis of the complex plane and $\zeta(\bar{s}) = \overline{\zeta}(s)$ when $\Re(s) > 1$.

By the same process, the Dirichlet eta function is expressed as follows:

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{[\cos(\beta \ln n) - i\sin(\beta \ln n)]}{n^{\alpha}}\,, \tag{8}$$

where $\Re(s) > 0$, and $\eta(1) = \ln(2)$ by continuity.

Using the same reasoning as above, the Dirichlet eta function is symmetrical with respect to the real axis of the complex plane. Furthermore, the factor $\left(1 - 2^{1-s}\right) = 1 - 2^{1-\alpha}\left(\cos(\beta \ln 2) - i\sin(\beta \ln 2)\right)$ also has mirror symmetry with respect to the real axis. As the product of holomorphic functions which are symmetrical with respect to the real axis yields a holomorphic function

which is also symmetrical with respect to the real axis, we get $\zeta(\bar{s}) = \overline{\zeta}(s)$ when $\Re(s) \in]0,1[$.

Let us consider the Riemann zeta functional (4), and introduce the function $\xi\colon \mathbb{C} \to \mathbb{C}$ such that:

$$\xi(s) = 2\Gamma(s)(2\pi)^{-s}\cos\left(\frac{s\pi}{2}\right). \tag{9}$$

If $\xi(s)$ is symmetrical with respect to the real axis, then by analytic continuation, the Riemann zeta function is also symmetrical with respect to the real axis, when $\Re(s) < 0$. From formula 6.1.23 p. 256 in [1], we have $\bar{\Gamma}(s) = \Gamma(\bar{s})$. Furthermorre, $(2\pi)^{-s} = \frac{\cos(\beta\ln 2\pi - i\sin(\beta\ln 2\pi))}{(2\pi)^{\alpha}}$, hence $(2\pi)^{-s}$ is symmetrical with respect to the real axis. From the formula 4.3.56 p. 74 in [1], we have $\cos\left(\frac{s\pi}{2}\right) = \cos\left(\frac{\alpha\pi}{2}\right)\cosh\left(\frac{\beta\pi}{2}\right) - i\sin\left(\frac{\alpha\pi}{2}\right)\sinh\left(\frac{\beta\pi}{2}\right)$. As $\cosh(-x) = \cosh(x)$ and $\sinh(-x) = -\sinh(x)$, $\cos\left(\frac{s\pi}{2}\right)$ is also symmetrical with respect to the real axis. Hence, $\xi(s)$ has mirror symmetry with respect to the real axis. Therefore, $\zeta(\bar{s}) = \overline{\zeta}(s)$ when $\Re(s) < 0$.

As the Riemann zeta function is a meromorphic function, by continuity $\zeta(\bar{s}) = \overline{\zeta}(s)$ on the lines $\Re(s) = 0$ and $\Re(s) = 1$.

Proposition 3 Given s a complex number and $\bar{s}$ its complex conjugate, we define a holomorphic function $\nu\colon \mathbb{C} \to \mathbb{C}$ such that:

$$\zeta(s) = \nu(s)\zeta(1-\bar{s}). \tag{10}$$

A property of $\nu(s)$ is that it is equal to one if $\Re(s) = 1/2$.

Proof Let us set $s = \alpha + i\,\beta$ where α and β are reals. When $\alpha = 1/2$, we get $\zeta(s) = \zeta\left(\frac{1}{2} + i\,\beta\right)$ and $\zeta(1-\bar{s}) = \zeta\left(\frac{1}{2} + i\,\beta\right)$. Hence, $\zeta(s) = \zeta(1-\bar{s})$ provided that $\Re(s) = \frac{1}{2}$. For a given $\alpha \neq \frac{1}{2}$, we have to introduce a complex factor $\nu(s)$ because we can no longer satisfy the two equations $\Re(\zeta(s)) = \Re(\zeta(1-\bar{s}))$ and $\Im(\zeta(s)) = \Im(\zeta(1-\bar{s}))$ with one degree of freedom given by β.

Proposition 4 Given s a complex number and $\bar{s}$ its complex conjugate, we have:

$$\frac{1}{\nu(s)} = 2\frac{\Gamma(\bar{s})}{(2\pi)^{\bar{s}}}\cos\left(\frac{\pi\bar{s}}{2}\right)\left[\frac{u^2 - v^2}{u^2 + v^2} + i\left(\frac{-2u\,v}{u^2 + v^2}\right)\right], \tag{11}$$

where $u = \Re(\zeta(s))$ and $v = \Im(\zeta(s))$.

Proof This formula is derived from the Riemann zeta functional. By the definition in *proposition 3*, we have $\frac{1}{\nu(s)} = \frac{\zeta(1-\bar{s})}{\zeta(s)}$. From (4), the Riemann zeta functional is expressed as $\zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^s}2\cos\left(\frac{\pi s}{2}\right)\zeta(s)$. Hence, $\zeta(1-\bar{s}) =$

$\frac{\Gamma(\bar{s})}{(2\pi)^{\bar{s}}} 2\cos\left(\frac{\pi\bar{s}}{2}\right)\zeta(\bar{s})$. From *proposition 2*, we have $\zeta(\bar{s}) = \overline{\zeta}(s)$, where $\overline{\zeta}(s)$ is the complex conjugate of $\zeta(s)$.
We use *proposition 1* and get $\zeta(\bar{s}) = \zeta(s)\left[\frac{u^2-v^2}{u^2+v^2} + i\left(\frac{-2u\,v}{u^2+v^2}\right)\right]$ where $u = \Re(\zeta(s))$ and $v = \Im(\zeta(s))$. Eq. (11) follows.

Proposition 5 Given s a complex number, the function $\nu(s)$ as defined above is equal to zero only at the points $s = 0, -2, -4, -6, -8, ..., -n$ where n is an even integer.

Proof We set $s = \alpha + i\,\beta$ where α and β are reals. The function $\nu(s)$ tends to zero, when its reciprocal $\frac{1}{\nu(s)}$ approaches $\pm\infty$. We use *proposition 4* to find the values when $\frac{1}{\nu(s)} \to \pm\infty$.

(i) We note that the term $\left[\frac{u^2-v^2}{u^2+v^2} + i\left(\frac{-2u\,v}{u^2+v^2}\right)\right]$ is bounded: we have $-1 \leqslant -\frac{v^2}{u^2+v^2} \leqslant \frac{u^2-v^2}{u^2+v^2} \leqslant \frac{u^2}{u^2+v^2} \leqslant 1$ and $-2 \leqslant \frac{-2\max(|u|,|v|)^2}{u^2+v^2} \leqslant \frac{-2uv}{u^2+v^2} \leqslant \frac{2\max(|u|,|v|)^2}{u^2+v^2} \leqslant 2$.

(ii) We note that the expression $\left[\frac{u^2-v^2}{u^2+v^2} + i\left(\frac{-2u\,v}{u^2+v^2}\right)\right]$ is never equal to zero as it is not possible to have both the real and imaginary parts equal to zero at the same time. For the real part to be equal to zero either $u = v$ or $u = -v$. When $u = v$ (or $u = -v$), the imaginary part is equal to -1 (or 1). For the imaginary part to be equal to zero, either u or v should be equal to zero. If u is equal zero (or v is equal to zero), then the real part is equal to -1 (or 1).

Eq. (5) can also be expressed as $\left[\frac{u^2-v^2}{u^2+v^2} + i\left(\frac{-2u\,v}{u^2+v^2}\right)\right] = \frac{u-i\,v}{u+i\,v}$. This formula is defined when $u + i\,v \neq 0$.

As implied by (i) and (ii), when $u + i\,v$ is in the neighborhood of 0, no matter how close to 0, the factor $\frac{u-i\,v}{u+i\,v}$ cannot take the value zero and is bounded.

Consequently, of (i) and (ii), $\nu(s) = 0$ at a point s_0 if only $\lim\limits_{s\to s_0} \cos\left(\frac{\pi\bar{s}}{2}\right)\Gamma(\bar{s}) = \pm\infty$.

We check that $\cos\left(\frac{\pi\bar{s}}{2}\right)$ is bounded. We have $\cos\left(\frac{\pi}{2}(\alpha - i\beta)\right) = \cos\left(\frac{\pi\alpha}{2}\right)\cosh\left(\frac{-\pi\beta}{2}\right) - i\,\sin\left(\frac{\pi\alpha}{2}\right)\sinh\left(\frac{-\pi\beta}{2}\right)$. As the cosine, the sine, the hyperbolic cosine and the hyperbolic sine are bounded on $]-\infty, \infty[$, $\cos\left(\frac{\pi\bar{s}}{2}\right)$ is bounded when β is finite.

Hence, the condition $\nu(s) = 0$ occurs only if $\Gamma(\bar{s}) \to \pm\infty$ or when the reciprocal Gamma $\frac{1}{\Gamma(\bar{s})} = 0$. The Euler form of the reciprocal Gamma function (see [2] p. 8) is $\frac{1}{\Gamma(s)} = s\prod_{n=1}^{\infty}\frac{1+\frac{s}{n}}{\left(1+\frac{1}{n}\right)^s}$. Thus, the only zeros of the reciprocal Gamma are $s = 0, -1, -2, -3, ..., -n$ where $n \in \mathbb{N}$. These are the points where $\nu(s)$ could potentially be equal to zero.

We check the points where the cosine term $\cos\left(\frac{\pi\bar{s}}{2}\right)$ is equal to zero. We have $\cos\left(\frac{\pi\bar{s}}{2}\right) = \cos\left(\frac{\pi}{2}(\alpha - i\beta)\right) = \cos\left(\frac{\pi\alpha}{2}\right)\cosh\left(\frac{-\pi\beta}{2}\right) - i\sin\left(\frac{\pi\alpha}{2}\right)\sinh\left(\frac{-\pi\beta}{2}\right)$. As the hyperbolic cosine is never equal to zero, the real part of $\cos\left(\frac{\pi\bar{s}}{2}\right)$ is equal to zero if and only if $\cos\left(\frac{\pi\alpha}{2}\right) = 0$. As we cannot have both the sine and cosine functions equal to zero simultaneously, when they share the same argument, the imaginary part of $\cos\left(\frac{\pi\bar{s}}{2}\right)$ equals zero if and only if $\sinh\left(\frac{-\pi\beta}{2}\right) = 0$, which occurs only if $\beta = 0$. Hence, the term $\cos\left(\frac{\pi\bar{s}}{2}\right)$ can be equal to zero on the real line at the points $s = \pm1, \pm3, \pm5, \pm7, ...$ Thus, $\nu(s)$ is equal to zero with certainty at the points $s = 0, -2, -4, -6, -8, ...$

What about the points $s = -1, -3, -5, -7, -9, ...$?

The Euler form of the Gamma function is expressed as $\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^s}{1+\frac{s}{n}}$. The Taylor series of $\cos\left(\frac{\pi s}{2}\right)$ in $-m$ where m is an odd integer is as follows: $\cos\left(\frac{\pi s}{2}\right) = \frac{\pi}{2}\sin\left(\frac{m\pi}{2}\right)(s+m) - \left(\frac{\pi}{2}\right)^3\sin\left(\frac{m\pi}{2}\right)(s+m)^3 + ...$. If we multiply the first term of the Taylor series by the m-th product of the Euler form of the Gamma function, we obtain a factor of order $\frac{s+m}{s+m}$, which is finite when s tends to $-m$. The multiplication of the successive terms of the Taylor series with the m-th product of the Euler form of the Gamma function leads to a factor of order $\frac{(s+m)^k}{s+m}$ where $k = 3, 5, 7, ...$, which converges towards zero when s tends to $-m$. Furtherore, the limit of the n-th product of the Euler form of the Gamma function $\frac{\left(1+\frac{1}{n}\right)^s}{1+\frac{s}{n}}$ when n tends to $+\infty$ is equal to 1. Therefore, $\lim_{s\to s_0}\cos\left(\frac{\pi\bar{s}}{2}\right)\Gamma(\bar{s})$ is finite at the points $s = -1, -3, -5, -7, -9, ...$ so we can say that $\nu(s) \neq 0$ at these points.

As a validation step, from *proposition 3* we have $\zeta(s) = \nu(s)\zeta(1-\bar{s})$, and from *proposition 9* the only pole of $\zeta(z)$ is at the point $z = 1$. Furthermore, according to section 3, we have $\zeta(1-\bar{s}) \neq 0$ when $\Re(s) < 0$. Hence, $\nu(s)$ cannot be equal to zero at the points $s = -1, -3, -5, -7, -9, ...$; otherwise, $\zeta(s)$ would be equal to zero at these points, which contradicts the non-trivial zeros of the Riemann zeta function (see section 4). Although $\nu(0) = 0$, $\zeta(0) \neq 0$ because the function $\zeta(z)$ has a pole at $z = 1$.

Proposition 6 Given s a complex number, the function $\nu(s)$ as defined above has an infinite number of poles at the points $s = 1, 3, 5, 7, 9, ..., n$ where n is an odd integer.

Proof We use the expression of the reciprocal of $\nu(s)$ introduced in *proposition 4*. We have shown in proof of *proposition 5*, that the factor $\left[\frac{u^2-v^2}{u^2+v^2} + i\left(\frac{-2u\,v}{u^2+v^2}\right)\right]$ cannot take the value zero and is bounded. Hence, a point s_0 is a pole of the function $\nu(s)$ if and only if $\lim_{s\to s_0}\Gamma(\bar{s})\cos\left(\frac{\pi\bar{s}}{2}\right) = 0$. Such poles occur either

when $\Gamma(\bar{s})$ or $\cos\left(\frac{\pi\bar{s}}{2}\right)$ are equal to zero. The term $\cos\left(\frac{\pi\bar{s}}{2}\right)$ is equal to zero at the points $s = \pm1, \pm3, \pm5, \pm7, ...$ The Euler form of the Gamma function is expressed as $\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^s}{1+\frac{s}{n}}$. As $\forall n \in \mathbb{N}^*$ the term $1+\frac{1}{n}$ is not equal to zero, we can say that the Gamma function is never equal to zero. However, the Gamma function tends to infinity at the points $s = 0, -1, -2, -3, ..., -n$ where $n \in \mathbb{N}$, hence the points $s = 1, 3, 5, 7, ...$ can be determined to be poles of $\nu(s)$ with certainty. The points $s = -1, -3, -5, -7, ...$ are poles of $\nu(s)$ only if the limit of $\Gamma(s)\cos\left(\frac{\pi s}{2}\right)$ is equal to zero when s tends to either of these points. The Taylor series of $\cos\left(\frac{\pi s}{2}\right)$ in $-m$ where m is an odd integer is as follows: $\cos\left(\frac{\pi s}{2}\right) = \frac{\pi}{2}\sin\left(\frac{\pi m}{2}\right)(s+m) - \left(\frac{\pi}{2}\right)^3\sin\left(\frac{\pi m}{2}\right)(s+m)^3 + ...$ If we multiply the m-th product of the Euler form of the Gamma function by the first term of the Taylor series we computed for $\cos\left(\frac{\pi s}{2}\right)$, we obtain a factor of order $\frac{s+m}{s+m}$, which does not converge towards zero when s tends to $-m$. The multiplication of the successive terms of the Taylor series with the m-th product of the Euler form of Gamma function leads to a factor of order $\frac{(s+m)^k}{s+m}$ where $k = 3, 5, 7, ...$, which converges towards zero when s tends to $-m$. In addition, the limit of the n-th product of the Euler form of the Gamma function $\frac{\left(1+\frac{1}{n}\right)^s}{1+\frac{s}{n}}$ when n tends to $+\infty$ is equal to 1. Hence, $\lim_{s\to s_0}\Gamma(s)\cos\left(\frac{\pi s}{2}\right) \neq 0$ at the points $s_0 = -1, -3, -5, -7, ...$ Thus, the points $s = -1, -3, -5, -7, ...$ are not within the set of poles of $\nu(s)$.

Proposition 7 Given s a complex number and $\bar{s}$ its complex conjugate, if s is a complex zero of the Riemann zeta function in the strip $\Re(s) \in]0, 1[$, then $1-\bar{s}$ must also be a zero.

Proof From *proposition 5* we have shown that $\nu(s)$ is only equal to zero at even negative integers including zero. Hence $\nu(s)$ is never reaches zero when $\Re(s) > 0$. Furthermore, in *proposition 6* we have shown that $\nu(s)$ has no poles in the strip $\Re(s) \in]0, 1[$. As $\zeta(s) = \nu(s)\zeta(1-\bar{s})$ from *proposition 4* and given that $\nu(s)$ has no poles nor zeros in the strip $\Re(s) \in]0, 1[$, we can say that if s is a zero, then $1-\bar{s}$ must also be a zero.

Proposition 8 Given s a complex number and $\bar{s}$ its complex conjugate, if s is a complex zero of the Riemann zeta function in the strip $\Re(s) \in]0, 1[$, then we have:

$$\zeta(s) = \zeta(1-\bar{s})\,. \tag{12}$$

Proof This is a corollary of *proposition 7*. If s and $1-\bar{s}$ are zeros of the Riemann zeta function, both terms $\zeta(s)$ and $\zeta(1-\bar{s})$ are equal to zero. Hence, both terms must match. The converse is not necessarily true, meaning that $\zeta(s) = \zeta(1-\bar{s})$ does not imply that s and $1-\bar{s}$ are zeros of the function.

Proposition 9 The only pole of the Riemann zeta function $\zeta(s)$ is a simple pole at $s = 1$.

This is documented in [16], p.13. A sketch ofthe proof is provided below for the sake of completleness.

Proof For notation purposes, we set $s = \alpha + i\,\beta$ where α and β are reals.

Using Abel's lemma for summation by parts, see [8] at p. 58, we get:

$$\sum_{n=1}^{m} n^{-s} = \sum_{n=1}^{m-1} n\left(n^{-s} - (n+1)^{-s}\right) + m^{1-s}\,, \tag{13}$$

where $m \in \mathbb{N}$.

When $\Re(s) > 1$, we have $\lim\limits_{m\to\infty} m^{1-s} = 0$. Thus, we get:

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-s} &= \sum_{n=1}^{\infty} n\left(n^{-s} - (n+1)^{-s}\right)\\ &= s\sum_{n=1}^{\infty} n\int_{n}^{n+1} x^{-s-1}\,dx\\ &= s\int_{1}^{\infty} \lfloor x\rfloor x^{-s-1}\,dx\\ &= \frac{s}{s-1} - s\int_{1}^{\infty}\{x\}x^{-s-1}\,dx\,,\end{aligned} \tag{14}$$

where $\lfloor x\rfloor$ denotes the floor function of x and $\{x\} = x - \lfloor x\rfloor$ denotes the fractional part of x.

Hence, we have:

$$\zeta(s) = \frac{s}{s-1} - s\int_{1}^{\infty}\{x\}x^{-s-1}\,dx\,, \tag{15}$$

where $\Re(s) > 1$.

To say that the only pole of $\zeta(s)$ on the half-plane $\Re(s) \geqslant 1$ is $s = 1$, we have too show that the integral $\int_1^\infty \{x\}x^{-s-1}\,dx$ in (15) is bounded when $\Re(s) \geqslant 1$.

Given a sequence u_i of complex numbers, we have the inequality $|\sum\limits_i u_i| \leqslant \sum\limits_i |u_i|$. As an integral is an infinite sum, this inequality still holds and we get:

$$\begin{aligned}\left|\int_1^{\infty}\{x\}x^{-s-1}\,dx\right| &\leqslant \int_1^{\infty}|\{x\}x^{-s-1}|\,dx\\ &\leqslant \int_1^{\infty}|\{x\}x^{(-\alpha-1)}e^{-i\,\beta\ln x}|\,dx\\ &\leqslant \int_1^{\infty}|\{x\}x^{-\alpha-1}|\,dx\\ &\leqslant \int_1^{\infty}\{x\}x^{-\alpha-1}\,dx\,.\end{aligned} \tag{16}$$

We have $\int_n^{n+1}\{x\}x^{-\alpha-1}\,dx < \int_n^{n+1}x^{-\alpha-1}\,dx$. Thus, we get:

$$\begin{aligned}\left|\int_1^{\infty}\{x\}x^{-s-1}\,dx\right| &< \sum_{n=1}^{\infty}\int_n^{n+1}x^{-\alpha-1}\,dx\\ &< \frac{1}{\alpha}\sum_{n=1}^{\infty}\left(n^{-\alpha}-(n+1)^{-\alpha}\right)\,.\end{aligned} \tag{17}$$

When $\alpha = 1$, we have:

$$\begin{aligned}\left|\int_1^{\infty}\{x\}x^{-s-1}\,dx\right| &< \sum_{n=1}^{\infty}\left(n^{-1}-(n+1)^{-1}\right)\\ &< \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\,.\end{aligned} \tag{18}$$

From the integral test [4] p. 132, we can show that the series $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)} < \sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ converges. Hence, the only pole on the line $\Re(s) = 1$ is at $s = 1$, which is a simple pole.

From the integral test, when $\alpha > 1$, the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ and $\sum\limits_{n=1}^{\infty}\frac{1}{(n+1)^{\alpha}}$ converge. Hence, $\left|\int_1^{\infty}\{x\}x^{-s-1}\,dx\right|$ in (18) is bounded. Thus, we can say that $\zeta(s)$ has no poles when $\Re(s) > 1$.

According to the Riemann zeta functional (4), we have:

$$\zeta(1-\alpha-\beta\,i) = \Gamma(\alpha+i\,\beta)(2\pi)^{-(\alpha+i\,\beta)}2\cos\left(\frac{(\alpha+i\,\beta)\pi}{2}\right)\zeta(\alpha+i\beta)\,. \tag{19}$$

The Euler form of the Gamma function is expressed as follows:

$$\Gamma(s) = \frac{1}{s} \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^s}{1+\frac{s}{n}} \,. \tag{20}$$

Hence, the Gamma function has no poles when $\Re(s) \geqslant 1$. The cosine term in (19) is bounded. Also, we have shown that the only pole of $\zeta(s)$ in the half-plan $\Re(s) \geqslant 1$ is at $s = 1$. The Riemann zeta function evaluated at 0 is $\zeta(0) = -\frac{1}{2}$, see [18] p. 135. Thus, considering (19) when $\alpha \geqslant 1$, we can say that $\zeta(s)$ has no poles when $\Re(s) \leqslant 0$.

The Dirichlet eta function is used as an extension of the Riemann zeta function to show that $\zeta(s)$ has no poles in the strip $\Re(s) \in]0, 1[$. As the factor $\left(1 - \frac{2}{2^s}\right)$ has no poles nor zeros in the strip $\Re(s) \in]0, 1[$, it is enough to show that the Dirichlet eta function is bounded, to say that $\zeta(s)$ has no poles when $\Re(s) \in]0, 1[$. The Dirichlet eta function in its integral form is expressed as follows:

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x+1}\, dx \,. \tag{21}$$

As the Gamma function is never equal to zero (see details in proposition 6), $\eta(s)$ is bounded if the integral $\int_0^{\infty} \frac{x^{s-1}}{e^x+1}\, dx$ is bounded. Let us consider the case when $\alpha \in]0, 1[$, hence:

$$\begin{aligned}
\left|\int_0^{\infty} \frac{x^{s-1}}{e^x+1}\, dx\right| &\leqslant \int_0^{\infty} \left|\frac{x^{s-1}}{e^x+1}\right| dx \\
&\leqslant \int_0^{\infty} \frac{x^{\alpha-1}}{e^x+1}\, dx \\
&< \int_0^{1} \frac{x^{\alpha-1}}{e^x+1}\, dx + \int_1^{\infty} \frac{1}{e^x+1}\, dx \\
&< \int_0^{1} \frac{x^{\alpha-1}}{e^x+1}\, dx + \int_1^{\infty} e^{-x}\, dx \,.
\end{aligned} \tag{22}$$

We solve $\int_0^{1} \frac{x^{\alpha-1}}{e^x+1}\, dx$ using integration by parts, where $\int u'v = [u\,v] - \int u\,v'$ with $u' = x^{\alpha-1}$ and $v = \frac{1}{e^x+1}$. We get:

$$\begin{aligned}
\int_0^{1} \frac{x^{\alpha-1}}{e^x+1}\, dx &= \left[\frac{x^{\alpha}}{\alpha}\frac{1}{e^x+1}\right]_0^1 + \int_0^1 \frac{x^{\alpha}}{\alpha}\frac{e^x}{(e^x+1)^2}\, dx \\
&< \left[\frac{x^{\alpha}}{\alpha}\frac{1}{e^x+1}\right]_0^1 + \int_0^1 \frac{1}{\alpha}\frac{e^x}{(e^x+1)^2}\, dx \\
&< \frac{1}{\alpha(e^1+1)} + \frac{1}{\alpha}\left(\frac{1}{2} - \frac{1}{e^1+1}\right) .
\end{aligned} \tag{23}$$

Thus, we have:

$$\left|\int_0^\infty \frac{x^{s-1}}{e^x+1}\,dx\right| < \frac{1}{\alpha(e^1+1)} + \frac{1}{\alpha}\left(\frac{1}{2} - \frac{1}{e^1+1}\right) + e^{-1}\,, \tag{24}$$

and $\eta(s)$ is bounded in the strip $]0,\,1[$, so we can say that $\zeta(s)$ has no poles in this strip.

3 Zeros when $\Re(s) > 1$

Though this is a familiar result, in this section the Euler product is used to prove there are no zeros when $\Re(s) > 1$. We also need the integral form of the remainder of the Taylor expansion of e^x.

The Taylor expansion of e^x in 0 may be expressed as follows:

$$e^x = 1 + x + \varepsilon(x)\,, \tag{25}$$

where $\varepsilon(x)$ is the remainder of the Taylor approximation.

The integral form of the remainder of the Taylor expansion of e^x is as follows:

$$\varepsilon(x) = \int_0^x (x-u)\,e^u du\,. \tag{26}$$

When $u \in [0,\,x]$, we have $(x-u)\,e^u \geqslant 0$. Hence, we have $\varepsilon(x) \geqslant 0$. Consequently, we get:

$$e^x \geqslant 1 + x\,. \tag{27}$$

Let us say p is a prime number larger than one, and $s = \alpha + i\,\beta$ a complex number where α and β are reals. We have:

$$\begin{aligned} |1 - p^{-s}| &\leqslant 1 + |p^{-s}| \\ &\leqslant 1 + |p^{-\alpha - i\,\beta}| \\ &\leqslant 1 + p^{-\alpha} \times |e^{(i\,\beta\ln p)}| \\ &\leqslant 1 + p^{-\alpha}\,. \end{aligned} \tag{28}$$

By setting $x = p^{-\alpha}$ in (27), we get:

$$1 + p^{-\alpha} \leqslant \exp\left(p^{-\alpha}\right)\,. \tag{29}$$

From (28) and (29), we get:

$$|1 - p^{-s}| \leqslant \exp\left(p^{-\alpha}\right)\,. \tag{30}$$

The Euler product [2] p. 6 is expressed as:

$$\zeta(s) = \prod_{p}^{\infty} \frac{1}{1 - p^{-s}}\,, \tag{31}$$

where p is the sequence of prime numbers larger than 1 and $\Re(s) > 1$.

Hence, we have:

$$|\zeta(s)| = \prod_{p}^{\infty} \frac{1}{|1 - p^{-s}|}\,. \tag{32}$$

Using inequality (30) in (32), we get:

$$\begin{aligned} |\zeta(s)| &\geqslant \prod_{p}^{\infty} \exp(-p^{-\alpha}) \\ &\geqslant \exp\left(-\sum_{p}^{\infty} p^{-\alpha}\right) \\ &> 0\,. \end{aligned} \tag{33}$$

We have $0 < \sum_{p}^{\infty} p^{-\alpha} < \sum_{n=1}^{\infty} n^{-\alpha}$. From the integral test, the series $\sum_{n=1}^{\infty} n^{-\alpha}$ converges when $\alpha > 1$, hence $\exp\left(-\sum_{p}^{\infty} p^{-\alpha}\right) > 0$. From (33), we get $\forall \alpha > 1, |\zeta(s)| > 0$. Hence, we can say the Riemann zeta function has no zeros when $\Re(s) > 1$.

4 Zeros for $\Re(s) < 0$

The below functional equation is used to obtain the zeros of the Riemann zeta function for $\Re(s) < 0$:

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)\,. \tag{34}$$

The derivation of this variant of the Riemann zeta functional is provided in [7] p. 8-12. By setting $s = \alpha + i\,\beta$ where α and β are reals, (34) is expressed as follows:

$$\zeta(\alpha + i\,\beta) = \pi^{(\alpha + i\,\beta - 1/2)} \frac{\Gamma\left(\frac{1-\alpha-i\,\beta}{2}\right)}{\Gamma\left(\frac{\alpha+i\,\beta}{2}\right)} \zeta(1 - \alpha - i\,\beta)\,. \tag{35}$$

Let us consider (35) when $\alpha < 0$. In the previous section, we have shown that the Riemann zeta function has no zeros when $\Re(s) > 1$, hence $\zeta(1 - \alpha - i\,\beta)$ is never equal to zero when $\alpha < 0$. The Gamma function is never equal to

zero (see insider of proof proposition 6). Thus, the zeros of the Riemann zeta function when $\Re(s) < 0$ are found at the poles of $\Gamma\left(\frac{s}{2}\right)$ provided $\zeta(1-s)$ has no poles when $\Re(s) < 0$. From *proposition 9*, the Riemann zeta function $\zeta(z)$ only has one pole at the point $z = 1$, hence $\zeta(1-s)$ has no poles when $\alpha < 0$. Thus, we obtain the trivial zeros of $\zeta(s)$ at $s = -2, -4, -6, -8, ...$ Note that $s = 0$ is not a zero of the Riemann zeta function due to the pole at $\zeta(1)$. We have $\zeta(0) = -\frac{1}{2}$, see [18].

5 Zeros on the lines $\Re(s) = 0, 1$

The fact that the Riemann zeta function has no zeros on the line $\Re(s) = 1$ was already established by both Hadamard and de la Vallée Poussin in their respective proofs of the prime-number theorem. A sketch of the proof of de la Vallée Poussin is available in [2] p. 79-80. We provide here a summary for the sake of completeness.

From the Euler product, we have:

$$\frac{1}{\zeta(s)} = \prod_{p}^{\infty}(1 - p^{-s})\,, \tag{36}$$

where p is the sequence of prime numbers larger than 1. Eq. (36) is defined for $\Re(s) > 1$.

By expanding (36), we get:

$$\frac{1}{\zeta(s)} = 1 - \sum_{p}^{\infty}(p^{-s}) + \sum_{p<q}^{\infty}(pq)^{-s} - \sum_{p<q<r}^{\infty}(pqr)^{-s} + ...\,, \tag{37}$$

where p, q, r, ... are prime numbers. Therefore, we get an infinite sum of integers, which are the product of unique primes. For such an integer n, the coefficient of n^{-s} is $+1$ if the number of prime factors of n is even, and -1 if the number of prime factors of n is odd. Thus, we get:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n)\, n^{-s}\,, \tag{38}$$

where $\mu(n)$ is the Möbius function.

We have $\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s} = \sum\limits_{n=1}^{\infty} e^{-s\ln(n)}$. Hence, we get:

$$\zeta'(s) = -\sum_{n=1}^{\infty} \ln(n)\, n^{-s}\,. \tag{39}$$

By multiplying (38) with (39) we get the logarithmic derivative of $\zeta(s)$, which is defined for $\Re(s) > 1$. The expansion yields:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty} \Lambda(n)\, n^{-s}\,, \tag{40}$$

where $\Lambda(n)$ is the Mangoldt function, which is equal to $\ln(p)$ if $n = p^k$ for some prime p and integer $k \geqslant 1$ and 0 otherwise.

Let us say we have a function f which is holomorphic in the neighborhood of a point a. Suppose there is a zero in a, hence we can write $f(z) = (z-a)^n\, g(z)$ where $g(a) \neq 0$. By computing the derivative of f, we get: $(z-a)\frac{f'(z)}{f(z)} = n + (z-a)\frac{g'(z)}{g(z)}$. We set $\varepsilon = z - a$, hence we get $\varepsilon\frac{f'(a+\varepsilon)}{f(a+\varepsilon)} = n + \varepsilon\frac{g'(a+\varepsilon)}{g(a+\varepsilon)}$. Suppose a is not a pole of f, hence we get $\frac{g'(a)}{g(a)}$ is equal to a constant. If we take the limit of the real part of $\varepsilon\frac{f'(a+\varepsilon)}{f(a+\varepsilon)}$ when ε tends to zero, we get:

$$w(f,a) = \lim_{\varepsilon\to 0} \Re\left[\varepsilon\, \frac{f'(a+\varepsilon)}{f(a+\varepsilon)}\right] = n\,, \tag{41}$$

which is the multiplicity of the zero in a.

Let us set $s = 1 + \varepsilon + i\,\beta$ where β is a real number and $\varepsilon > 0$. We get:

$$\Re\left[\varepsilon\frac{\zeta'(s)}{\zeta(s)}\right] = -\varepsilon\sum_{n=2}^{\infty} \Lambda(n)\, n^{-1-\varepsilon} \cos(\beta\, \ln(n))\,. \tag{42}$$

The proof uses the Mertens' trick, which is based on the below inequality:

$$\begin{aligned} 0 \leqslant (1+\cos\theta)^2 &= 1 + 2\cos\theta + \cos^2\theta \\ &= 1 + 2\cos\theta + \frac{1+\cos(2\theta)}{2} \\ &= \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos(2\theta)\,. \end{aligned} \tag{43}$$

Hence, we get:

$$3 + 4\cos\theta + \cos(2\theta) \geqslant 0\,. \tag{44}$$

From (42) and (43), we get:

$$3\,\Re\left[\varepsilon\frac{\zeta'}{\zeta}(1+\varepsilon)\right] + 4\,\Re\left[\varepsilon\frac{\zeta'}{\zeta}(1+\varepsilon+i\,\beta)\right] + \Re\left[\varepsilon\frac{\zeta'}{\zeta}(1+\varepsilon+2\,i\,\beta)\right] \leqslant 0\,, \tag{45}$$

where $\varepsilon > 0$. By taking the limit of (45) when ε tends to zero from the right ($\varepsilon > 0$), we get:

$$3\,w(\zeta,1) + 4\,w(\zeta,1+i\,\beta) + w(\zeta,1+2\,i\,\beta) \leqslant 0\,. \tag{46}$$

From *proposition 9*, $\zeta(s)$ only has a simple pole in $s = 1$. Hence, when $\beta \neq 0$, we have $w(\zeta, 1 + 2\,i\,\beta) \geqslant 0$ and $w(\zeta, 1) = -1$. Thus, we get:

$$\begin{aligned} 0 \leqslant w(\zeta, 1 + i\,\beta) &\leqslant -\frac{3}{4} w(\zeta, 1) \\ &\leqslant \frac{3}{4}\,. \end{aligned} \tag{47}$$

As the Riemann zeta function is holomorphic, the multiplicity of a zero in a denoted $w(\zeta, a)$ must be an integer. Thus, according to (47), $\forall \beta \neq 0, w(\zeta, 1 + i\,\beta) = 0$. Hence, we can say that the Riemann zeta function has no zeros on the line $\Re(s) = 1$.

The fact that $\zeta(s)$ has no zeros on the line $\Re(s) = 0$ is obtained by reflecting the line $\Re(s) = 1$ with the Riemann zeta functional. By considering (35) when $\alpha = 0$, we get:

$$\zeta(i\,\beta) = \pi^{(i\,\beta - 1/2)} \frac{\Gamma\left(\frac{1 - i\,\beta}{2}\right)}{\Gamma\left(\frac{i\,\beta}{2}\right)} \zeta(1 - i\,\beta)\,. \tag{48}$$

As the Gamma function is never equal to zero, the zeros on the line $\Re(s) =$ 0require that $\Gamma\left(\frac{i\,\beta}{2}\right)$ is a pole. The only pole of $\Gamma\left(\frac{i\,\beta}{2}\right)$ occurs when $\beta = 0$. However, as $\zeta(s)$ has a pole in $s = 1$, we cannot say that the point $s = 0$ is a zero. In fact, we have $\zeta(0) = -\frac{1}{2}$. To show that $\zeta(0) = -\frac{1}{2}$, we use the Riemann zeta functional (3), which is expressed as follows:

$$\zeta(s) = \pi^{s-1} 2^s \sin\left(\frac{s\pi}{2}\right) \Gamma(1 - s) \zeta(1 - s)\,. \tag{49}$$

The Gamma functional equation, which is derived using integration by part on the integral form of the Gamma function, see formula 6.1.15 p. 256 in [1] is expressed as follows:

$$s\Gamma(s) = \Gamma(s + 1)\,. \tag{50}$$

By multiplying (49) with $(1 - s)$, we get $(1 - s)\zeta(s) = \pi^{s-1} 2^s \sin\left(\frac{s\pi}{2}\right)(1 - s)\Gamma(1 - s)\zeta(1 - s)$. From (50), we get $(1 - s)\Gamma(1 - s) = \Gamma(2 - s)$. As $\zeta(s)$ has a simple pole in $s = 1$ (see *proposition 9*), we get $\lim_{s\to 1}(1 - s)\Gamma(s) = -1$. Also, we have $\lim_{s\to 1} \pi^{s-1} 2^s \sin\left(\frac{s\pi}{2}\right) \Gamma(2 - s)\zeta(1 - s) = 2\zeta(0)$. Hence, $\zeta(0) = -\frac{1}{2}$. Therefore, we can say the Riemann zeta function has no zeros on the line $\Re(s) = 0$.

6 Zeros in the critical strip $\Re(s) \in]0, 1[$

Let us set the complex number $s = \alpha + i\,\beta$ where α and β are reals and its complex conjugate $\bar{s} = \alpha - i\,\beta$. From *proposition 8*, if s is a zero of the Riemann zeta function in the strip $\Re(s) \in]0,\,1[$, then $\zeta(s) = \zeta(1 - \bar{s})$.

Let us rewrite the latter equation in terms of α and β. We get:

$$\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta)\,. \tag{51}$$

A solution set of (51) is $\alpha = \frac{1}{2}$ for any $\beta \in \mathbb{R}$, where zeros of the Riemann zeta function are found.

The Riemann hypothsis states that all non-trivial zeros, i.e. in the critical strip $\Re(s) \in]0, 1[$ lie on the critical line $\Re(s) = 1/2$. By reflexion of the zeros of the Riemann zeta function with respect to the critical line $\Re(s) = \frac{1}{2}$ (see *proposition 8*), it is enough to show there are no zeros on either of the intervals $\Re(s) \in]0, \frac{1}{2}[$ or $]\frac{1}{2}, 1[$, to say all non-trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = 1/2$.

7 Discussion

The Riemann hypothesis is an old problem of number theory related to the prime-number theorem, which statement is that all non-trivial zeros lie on the critical line $\Re(s) = 1/2$. While the Riemann zeta functional leads to *proposition 8*, this criterion alone is not enough to say whether the Riemann hypothesis is true or false. Since this criterion establishes a symmetry between the zeros of the Riemann zeta function on either sides of the critical line $\Re(s) = 1/2$, it is enough to show that the Riemann zeta function has no zeros on either of the strips $\Re(s) \in]0, \frac{1}{2}[$ or $]\frac{1}{2}, 1[$, to say the Riemann hypothesis is true.

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